$\mathcal{D}$-ELLIPTIC SHEAVES AND UNIFORMISATION

LENNY TAELMAN

1. Introduction

This text seeks to expose a function field analogue - in the sense of $\mathbb{F}_p[t]$ versus $\mathbb{Z}$ - to objects such as modular curves and quaternionic Shimura curves.

Over the field of complex numbers, these curves permit a dual description as either algebraic varieties or Riemann surfaces. The former is obtained as a parameter space for certain Abelian varieties, the latter as a quotient space of a homogeneous space, to wit the complex upper half plane, by a discrete group action.

Modelled after the classical modular curves are the Drinfel'd modular curves and the higher dimensional Drinfel'd modular varieties. Similarly, they can be treated either algebraically or analytically (see [3]). A wider class of varieties, containing at once the Drinfel'd modular varieties and the proper counterparts of the quaternionic Shimura curves, consists of the moduli spaces of Laumon, Rapoport and Stuhler. They are algebraic varieties that parametrise certain algebraic objects, namely $\mathcal{D}$-elliptic sheaves (see [7]). However, the dual, analytic, description is absent from the literature.

Thus, in a slightly more precise formulation, this manuscript aims to give the moduli spaces for $\mathcal{D}$-elliptic sheaves an analytic description as quotients of homogeneous spaces. This proceeds roughly in three steps. First, $\mathcal{D}$-elliptic sheaves are shown to form a particular subclass of the category of Anderson t-motives (§2, proposition 2.13). Second, these Anderson t-motives are proven to be *uniformisable*, meaning that they can be obtained as quotients of vector spaces by lattices, and the class of lattices so obtained is determined (§3, propositions 3.3 and 3.5). Finally, these lattices are parametrised by quotients of homogeneous spaces; this yields the main result (§4, theorem 4.8).

This uniformisation of the varieties of Laumon, Rapoport and Stuhler at the "infinite" valuation complements recent work of Hausberger ([5]). In the spirit of Drinfel'd and Cherednik's p-adic uniformisation of Shimura curves, he treats the analytic structure at the "ramified"

places. Hausberger has confided to the author that he has also obtained, but not published, a uniformisation at the "infinite" valuation.

2. $\mathcal{D}$-elliptic sheaves and Anderson A-motives

standard Drinfeld data

2.1. The following data are fixed. A complete, smooth and geometrically irreducible curve X over a finite field $\mathbb{F}_q$ of characteristic p and a closed point $\infty \in X$ called infinity. The degree of the residue class field of ∞ over $\mathbb{F}_q$ is denoted by α. The function field of X is denoted by $F = \mathbb{F}_q(X)$ and the subring of functions which are regular everywhere except possibly at infinity by $A = \mathrm{H}^0(X - \infty, \mathcal{O}_X)$. On A, deg is defined as the pole order at infinity. The completion of F with respect to the infinite valuation is F_∞. Also, a complete and algebraically closed field $\mathbf{C}$ containing F_∞ is fixed and the resulting embeddings denoted by $\imath : F_\infty \to \mathbf{C}$, $\imath : F \to \mathbf{C}$ and $\imath : A \to \mathbf{C}$. Of course, the kernel of $\imath : A \to \mathbf{C}$ is trivial, this corresponds to what is usually called "infinite characteristic", and reflects the particular interest into analytic structures taken in this exposition. Put differently: reductions modulo primes of A will not be touched upon. Finally, denote by $\hat{A}$ the product of all completions A_v over all places $v \neq \infty$ and by $\mathbb{A}_A = \hat{A} \otimes_A F$ the adèle ring away from infinity.

the central simple algebra

2.2. Also, a central simple F-algebra D, unramified at ∞ is fixed. Denote the dimension of D over its centre by d^2. Let L be an algebraically closed field containing $\mathbb{F}_q$, then $D \otimes_{\mathbb{F}_q} L \cong D \otimes_F L(X)$ is isomorphic to the full matrix algebra $\mathrm{M}(d, L(X))$, since $L(X)$ is a C_1 field.

Fix a subsheaf of $\mathcal{O}_X$-algebras $\mathcal{D}$ inside the constant sheaf D with the property that for all points $x \in X$, $\mathcal{D}_x$ is a maximal order in D_x (it is shown in [7] that such sheaves exist.) Then $O_D := \mathrm{H}^0(X - \infty, \mathcal{D})$ is a maximal A-order in D.

the sheaves

2.3. Let B be a $\mathbf{C}$-algebra and denote $X \times_{\mathbb{F}_q} \mathrm{Spec}(B)$ by X_B. The curve X_B on $\mathrm{Spec}(B)$ comes equipped with a section $\imath^\# : \mathrm{Spec}(B) \to X_B$ which derives from the embedding $\imath$ of the function field of X into $\mathbf{C}$ and takes as constant value the generic point of X.

The projections $X_B \to X$ and $X_B \to \mathrm{Spec}(B)$ are denoted by pr_X and pr_S respectively. The pullback of the geometric q-Frobenius on $\mathrm{Spec}(B)$ to X_B by pr_X by σ. Thus σ acts trivially on X and as the q-th power endomorphism on the ring of functions B.

2.4. **Definition.** *A* $\mathcal{D}$-elliptic sheaf *on* X_B *is a commutative diagram:*

$$\begin{tikzcd} \dots \arrow[r, "j"] & \mathcal{E}_{-1} \arrow[r, "j"] & \mathcal{E}_0 \arrow[r, "j"] & \mathcal{E}_1 \arrow[r, "j"] & \mathcal{E}_2 \arrow[r, "j"] & \dots \\ \dots \arrow[r, "j"] & \sigma^*\mathcal{E}_{-2} \arrow[r, "j"] \arrow[u, "t"] & \sigma^*\mathcal{E}_{-1} \arrow[r, "j"] \arrow[u, "t"] & \sigma^*\mathcal{E}_0 \arrow[r, "j"] \arrow[u, "t"] & \sigma^*\mathcal{E}_1 \arrow[r, "j"] \arrow[u, "t"] & \dots \end{tikzcd}$$

where the $\mathcal{E}_i$ *are locally free right* $\mathrm{pr}_X^*\mathcal{D}$*-modules of rank* 1 *and the* t *and the* j *are* $\mathrm{pr}_X^*\mathcal{D}$*-linear injections. These data are constraint to the following conditions:*

(1) Periodicity*: the composition of* $d\alpha$ *consecutive* j*'s defines an identification* $\mathcal{E}_{i+d\alpha} = \mathcal{E}_i \otimes_{\mathcal{O}_{X_B}} (\mathcal{O}_X(\infty) \boxtimes \mathcal{O}_{\mathrm{Spec}(B)})$ *through the natural embedding* $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(\infty)$*;*
(2) Pole*: the direct image of* $\mathcal{E}_i/j(\mathcal{E}_{i-1})$ *by* pr_B *is a locally free* $\mathcal{O}_{\mathrm{Spec}(B)}$*-module of rank* d*;*
(3) Zero*:* $\mathcal{E}_i/t(\sigma^*\mathcal{E}_{i-1})$ *is the direct image by* $\imath^\# : \mathrm{Spec}(B) \to X_B$ *of a locally free* $\mathcal{O}_{\mathrm{Spec}(B)}$*-module.*

A shorthand notation for such an object is $(\mathcal{E}_i, j, t)$*. A* morphism of $\mathcal{D}$-elliptic sheaves *on* X_B *is a collection of* $\mathrm{pr}_X^*\mathcal{D}$*-linear morphisms* $\mathcal{E}_i \to \mathcal{E}'_i$ *that commute with the* j*'s and the* t*'s.*

décalage

2.5. It will be necessary to consider elliptic $\mathcal{D}$-sheaves up to *décalage*. Formally this can be done by allowing more morphisms. Namely, a morphism of $\mathcal{D}$-elliptic sheaves up to décalage consists of a fixed integer n and a collection of $\mathrm{pr}_X^*\mathcal{D}$-linear morphisms $\mathcal{E}_i \to \mathcal{E}'_{i+n}$ commuting with the j's and t's.

the modules

2.6. **Definition.** *An Anderson A-motive with multiplication by* O_D *over* $\mathbf{C}$ *is a left* $O_D^{\mathrm{op}} \otimes_{\mathbb{F}_q} \mathbf{C}[\sigma]$*-module* M *satisfying*

(1) M *is projective of rank* 1 *over* $O_D^{\mathrm{op}} \otimes \mathbf{C}$
(2) M *is projective of rank* d *over* $\mathbf{C}[\sigma]$
(3) *there exists a positive integer* n *such that* $(a - \imath(a))^n(M/\sigma M) = 0$ *for all* $a \in A$*.*

2.7. Denote by σ the endomorphism of $O_D^{\mathrm{op}} \otimes_{\mathbb{F}_q} \mathbf{C}$ that acts trivially on O_D^{op} and as the q-th power map on $\mathbf{C}$. Then an Anderson A-motive with multiplication by O_D can also be described as a $O_D^{\mathrm{op}} \otimes \mathbf{C}$-module M on which the action of σ is described by either a linear morphism $\tau : \sigma^*M \to M$ or a semi-linear map from M to itself, also (abusively) denoted by $\tau : M \to M$.

As is already suggested in [7, §3], there is a close relationship between $\mathcal{D}$-elliptic sheaves and Anderson t-motives (in this manuscript called A-motives). This kinship is best understood through Drinfel'd's

theory of vector bundles on the non-commutative projective line. For more details on these objects, one can consult §3 of *loc. cit.*, which establishes most of what is needed in order to properly relate Anderson A-motives with multiplication by O_D to $\mathcal{D}$-elliptic sheaves. In the following paragraphs the remaining steps are taken.

from sheaves to modules

2.8. Since by the pole axiom, the subsequent quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ in a $\mathcal{D}$-elliptic sheaf $(\mathcal{E}_i, j, t)$ on $X_{\mathbf{C}}$ are supported at ∞, the $O_D^{\mathrm{op}} \otimes_k \mathbf{C}$-module $M := \mathrm{H}^0(X_{\mathbf{C}} - \infty, \mathcal{E}_i)$ is independent of i. Therefore the $t : \sigma^*\mathcal{E}_i \to \mathcal{E}_{i+1}$ induce a morphism $\tau : \sigma^* M \to M$ which turns the $O_D^{\mathrm{op}} \otimes_k \mathbf{C}$-module M into a $O_D^{\mathrm{op}} \otimes_k \mathbf{C}[\sigma]$-module (as in 2.7). By construction, M depends only on the given $\mathcal{D}$-elliptic sheaf up to décalage. It follows from [7, 3.17] that M is an Anderson A-motive with multiplication by O_D.

from modules to sheaves

2.9. The above construction is functorial and it is possible to define an inverse functor. It suffices to extend the given $\mathbf{C}[\sigma]$-module M to a vector bundle on the non-commutative projective line with coordinate σ and then apply the equivalence 3.17 of *loc. cit.* This is tantamount to a module-theoretic construction, that will be described next, first in the rank one Drinfel'd module case, that is, for $D = F$, and then for a general D.

2.10. Let $D = F$. The non-commutative ring of power series in σ^{-1}, denoted by $\mathbf{C}[[\sigma^{-1}]]$, has the skew field of Laurent series $\mathbf{C}((\sigma^{-1}))$ as quotient field. Given an M, a Anderson A-motive with multiplication by $O_D = A$, under the aforementioned equivalence, the definition of a $\mathcal{D}$-elliptic sheaf $(\mathcal{E}_i, j, t)$ such that $M = \mathrm{H}^0(X_{\mathbf{C}} - \infty, \mathcal{E}_i)$ boils down to the definition of a free rank one $\mathbf{C}[[\sigma^{-1}]]$-submodule

$$W \subset \mathbf{C}((\sigma^{-1})) \otimes_{\mathbf{C}[\sigma]} M.$$

The right-hand side of this inclusion is isomorphic to $\mathbf{C}((\sigma^{-1}))$, and the corresponding submodules are precisely those generated by a σ^m for an integer m. It can be checked that every such submodule defines a vector bundle on the non-commutative projective line with the correct properties, hence also an elliptic sheaf. Different choices of m define elliptic sheaves that are equivalent under décalage.

Morita equivalence

2.11. Extending the above reasoning to the general case is a matter of applying Morita equivalence. As this equivalence is also central to several of the proofs yet to come, it is worthwhile to recall a precise formulation here. Let R be a ring with unity. Then to every R-module N it is possible to associate the $\mathrm{M}(n, R)$-module $N \oplus N \oplus \cdots \oplus N$, the direct sum of n copies of N. *Morita equivalence* states that this defines an equivalence from the category of R-modules to the category of

$\mathrm{M}(n, R)$-modules, and that this equivalences preserves such properties as projectivity, injectivity, and finite generation. Also, the construction is functorial in R. A detailed exposition can be found in [6, §7].

from modules to sheaves (continued)

2.12. The restriction on D is now dismissed. Now $\mathbf{C}((\sigma^{-1})) \otimes_{\mathbf{C}[\sigma]} M$ is a module over $F_\infty \otimes_A O_D^{\mathrm{op}}$ and the submodule

$$W \subset \mathbf{C}((\sigma^{-1})) \otimes_{\mathbf{C}[\sigma]} M$$

is required to be isomorphic to $\mathbf{C}[[\sigma^{-1}]]^d$ and to be equipped with a continuous action by $\mathcal{D}_\infty \subset F_\infty \otimes_A O_D^{\mathrm{op}}$. Since $\mathcal{D}_\infty$ is a matrix algebra inside the matrix algebra $F_\infty \otimes_A O_D^{\mathrm{op}}$, the machinery of Morita applies. It turns the set of such W into the set of $\mathbf{C}[[\sigma^{-1}]]$-submodules of $\mathbf{C}((\sigma^{-1}))$. As above, these are generated by some σ^m and different m define different representatives of the same class of $\mathcal{D}$-elliptic sheaves up to décalage.

The above discussion (2.8-2.12) sums up to:

2.13. **Proposition.** *The category of $\mathcal{D}$-elliptic sheaves on $X_{\mathbf{C}}$ up to décalage is equivalent to the category of Anderson A-motives with multiplication by O_D.*

Drinfel'd modules

2.14. **Example.** If O_D is the algebra of $d \times d$ matrices over A then Morita equivalence implies the equivalence of Anderson A-motives with multiplication by O_D and Anderson A-motives of dimension 1 and rank d. Thus the theory of $\mathcal{D}$-elliptic sheaves includes the Drinfel'd modules as a special case.

level structures

2.15. Given an effective divisor I on $X - \infty$ it is possible to define level I-structures on A-motives with multiplication by O_D as well as on $\mathcal{D}$-elliptic sheaves such that the equivalence between the two categories extends to an equivalence of the categories "with level structure". For the sake of clarity, the results in this text are first stated and proved ignoring level structures, and the corresponding propositions with level structure are simply stated as corollaries. The omitted proofs are never hard.

The I in the succeeding definitions refers in *abus de notation* to both the divisor I on X and the corresponding ideal $I \subset A$.

level structures on sheaves

2.16. Since ∞ and I are disjoint, the restriction $\mathcal{E}_I$ of the X_B-sheaf $\mathcal{E}_i$ to I_B is independent of i and for the same reason the injection t restricts to an isomorphism $t : \sigma^*\mathcal{E}_I \to \mathcal{E}_I$. A *level I-structure on a $\mathcal{D}$-elliptic sheaf* $(\mathcal{E}_i, j, t)$ is an isomorphism of $\mathcal{O}_{I_B}$-sheaves: $\mathcal{D}_I \boxtimes \mathcal{O}_B \to \mathcal{E}_I$ such that the Frobenius $\sigma^*\mathcal{O}_B \to \mathcal{O}_B$ corresponds to $t : \sigma^*\mathcal{E}_I \to \mathcal{E}_I$.

level structures on modules

2.17. A *level I structure on an Anderson A-motive with multiplication by O_D* is the choice of an isomorphism $(O_D^{\mathrm{op}}/IO_D^{\mathrm{op}}) \otimes_{\mathbb{F}_q} \mathbf{C} \to M/IM$ such that the Frobenius on $\mathbf{C}$ matches the action of σ on M/IM.

3. Uniformisation of the Anderson A-motives

introduction

Whereas all Abelian varieties over the field of the complex numbers can be obtained as the quotient of a complex vector space by a lattice, the parallel statement for Anderson A-motives is false in general. Those Anderson A-motives that *can* be obtained in such a way are called *uniformisable*. It will be shown in this section that the A-motives considered above, *videlicet* the A-motives with multiplication by O_D, are in fact uniformisable.

on uniformisability

3.1. A very brief overview of the results of [1] is given. There is an anti-equivalence of categories that assigns to an Anderson A-motive M of dimension d, an action of A on an algebraic group $E(M) \cong \mathbb{G}_a^d$, satisfying certain properties. This $E(M)$ occurs naturally in an exact sequence of *analytic* A-modules:

$$\Lambda \longrightarrow \mathrm{Lie}(E(M)) \xrightarrow{\exp_{E(M)}} E(M)$$

Here $\mathrm{Lie}(E(M))$ is a d-dimensional vector space over $\mathbf{C}$ on which A acts by the embedding $\imath : A \hookrightarrow \mathbf{C}$, and Λ is a lattice, *i.e.* a discrete and projective A-submodule. The Anderson A-motive M can be recovered from this lattice if exp is surjective. There exist, however, M for which this is not the case. M is called *uniformisable* when $\exp_{E(M)}$ *is* surjective and by [1, §2] this is the case precisely when the rank of Λ equals the rank of M (in general $\mathrm{rk}_A(\Lambda) \leq \mathrm{rk}_{A\otimes\mathbf{C}}(M)$).

a criterion for uniformisability

3.2. Fix a subring $\mathbb{F}_p[a] \subset A$ such that a has a pole of order prime to p at ∞ and such that the resulting fraction field extension $F/\mathbb{F}_p(a)$ is separable. If the rank of A over $\mathbb{F}_p[a]$ is r, then an Anderson A-motive with multiplication by O_D is an Anderson $\mathbb{F}_p[a]$-motive of rank rd^2 and dimension d. Given an Anderson A-motive over $\mathbf{C}$ it is now possible to extend its scalars from $\mathbf{C}[a]$ to $\mathbf{C}\{\{a\}\}$ - the ring of power series with convergence radius at least 1 - which yields a module $\tilde{M} = M \otimes_{\mathbf{C}[a]} \mathbf{C}\{\{a\}\}$. The semilinear action $\tau : M \to M$ extends to a semilinear $\tilde{\tau} : \tilde{M} \to \tilde{M}$ and M is said to be *analytically trivial* if $\tilde{M}$ has a $\mathbf{C}\{\{a\}\}$-basis consisting of $\tilde{\tau}$-invariant vectors. This is all that is needed to state Anderson's criterion for uniformisability [1, §2]. Namely, it states that M is uniformisable if and only if it is analytically trivial.

uniformisability of the modules

3.3. **Proposition.** *Over* $\mathbf{C}$*, all* d*-dimensional* A*-motives with multiplication by* O_D *are uniformisable.*

Proof. Fix an isomorphism $O_D^{\mathrm{op}} \otimes_{\mathbb{F}_q} \mathbf{C} \cong \mathrm{M}(d, A \otimes_{\mathbb{F}_q} \mathbf{C})$.

Let M be an A-motive as in the proposition. Then M is a projective $O_D^{\mathrm{op}} \otimes \mathbf{C}$-module of rank one together with a semilinear map $\tau : M \to M$ commuting with the action of O_D^{op} and having a determinant equal to $(a - \imath(a))^d$ up to a unit. Morita equivalence associates to these data a module N of rank n/d over $A \otimes \mathbf{C}$ equipped with a semilinear $\tau_N : N \to N$ of determinant a unit times $(a - \imath(a))$ such that M can be recovered from N as a d-fold direct sum: $M \cong N \oplus N \oplus \cdots \oplus N$. The A-motive N has dimension 1, thus determines a Drinfel'd module. Drinfel'd modules are uniformisable ([3]), and it follows from Anderson's analytic triviality criterion that N is analytically trivial. This shows that $\tilde{M} = \tilde{N} \oplus \tilde{N} \oplus \cdots \oplus \tilde{N}$ has a $\tilde{\tau}$-invariant $\mathbf{C}\{\{a\}\}$-basis, hence M is uniformisable. □

lattices and representations

3.4. Thus, over $\mathbf{C}$, to every d-dimensional A-motive M with multiplication by O_D, or to every $\mathcal{D}$-elliptic sheaf on $X_{\mathbf{C}}$, a discrete lattice Λ of A-rank r^2 in $V := \mathrm{Lie}(E(M))$ is assigned. This assignment is a faithful functor on the category of uniformisable A-motives [1, 2.12.2], and this immediately implies that $\Lambda \subset V$ is more than just an A-lattice in a $\mathbf{C}$-vector space: it naturally carries the structure of an O_D-lattice in a faithful $\mathbf{C}$-linear representation of D. The following proposition confirms that every pair $\Lambda \subset V$ of a rank one O_D-lattice in a faithful d-dimensional representation of D occurs in such a way.

surjectivity of the construction

3.5. **Proposition.** *Let* $\Lambda \subset V$ *be a discrete projective and rank one* O_D*-submodule of a* d*-dimensional faithful linear representation of* D*. There exists a uniformisable Anderson* A*-motive with multiplication by* O_D *and an isomorphism* $\mathrm{Lie}(E(M)) \to V$ *that surjects* $\ker(\exp_{E(M)})$ *onto* Λ*.*

Proof. By [1, corollary 3.5.1], it is sufficient to find an A-lattice $\Lambda' \subset V$ whose F_∞-span equals the F_∞-span of Λ and which is the kernel of the exponential function of some uniformisable abelian A-module. Since D is unramified at the infinite place, $D \otimes F_\infty$ is isomorphic to $\mathrm{M}(d, F_\infty)$, hence the F_∞-span of Λ splits under the action of the diagonal idempotents as a direct sum of d terms each lying within a one-dimensional linear subspace of V. Each such term is the span of a suitable lattice in a one-dimensional linear space and the direct sum of these lattices is the Λ' which is being sought for. It is the kernel of the exponential

function of an abelian A-module which is uniformisable because it is the direct sum of d suitable Drinfel'd modules. $\square$

3.6. **Corollary.** *The category of Anderson A-motives with multiplication by O_D is anti-equivalent with the category of pairs (V, Λ), where V is a faithful d-dimensional representation of D and $\Lambda \subset V$ an O_D-lattice.*

4. Analytic moduli spaces of $\mathcal{D}$-elliptic sheaves

In this section, all $\mathcal{D}$-elliptic sheaves are considered up to décalage (see 2.5).

The requisite preparations have now been made in order to give the varieties of Laumon, Rapoport and Stuhler an analytic description. The main ingredient is the classification of the pairs (V, Λ) of corollary 3.6. These are classified by quotients of the Drinfel'd symmetric space $\Omega^d(\mathbf{C}) = \mathbb{P}^{d-1}(\mathbf{C}) - \mathbb{P}^{d-1}(F_\infty)$, in very much the same way that some Shimura varieties are constructed as quotients of the complex upper half plane. In what follows $D^*(R)$ denotes $(D \otimes_F R)^*$, for an F-algebra R.

classification of lattices

4.1. **Proposition.** *There is a natural bijection between the set of isomorphism classes of pairs (V, Λ) of d-dimensional $\mathbf{C}$-linear representations of D with O_D-lattice $\Lambda \subset V$ and the double co-sets in*

$$D^* \backslash \left[\Omega^d(\mathbf{C}) \times D^*(\mathbb{A}_A)/U\right]$$

where D^* acts diagonally on the product and in particular on $\Omega^d(\mathbf{C})$ by the choice of an isomorphism $D \otimes_F F_\infty \cong \mathrm{M}(d, F_\infty)$ and where U is the compact open subgroup $(O_D \otimes_A \hat{A})^* \subset D^*(\mathbb{A}_A)$.

Proof. Start of with a pair (V, Λ). Since all representations are conjugate, it is possible to assume without loss of generality that $V = F_\infty^d \otimes \mathbf{C} = \mathbf{C}^d$ on which D acts by a fixed isomorphism $D \otimes_A F_\infty \cong \mathrm{M}(d, F_\infty)$. Consider the F-span $F\Lambda$ of the lattice. This is a free module of rank 1 over D lying inside V and the choice of a generator marks a point on $\mathbb{P}^{d-1}(\mathbf{C})$ and identifies $F\Lambda$ with D. The marked point lies in $\Omega^d(\mathbf{C}) \subset \mathbb{P}^{d-1}(\mathbf{C})$ by the discreteness of Λ. The embedding $\Lambda \subset D$ can be tensored to an embedding $\hat{A}\Lambda \subset D \otimes \mathbb{A}_A$ and the former can be recovered from the latter as $\Lambda = \hat{A}\Lambda \cap D \otimes 1$. But all projective $O_D \otimes \hat{A}$-modules are free, consequently the projective O_D-submodules $\Lambda \subset D$ of rank one are in bijection with the free rank one $O_D \otimes \hat{A}$-submodules of $D \otimes \mathbb{A}_A$ and the latter are in bijection with $(D \otimes \mathbb{A}_A)^*/(O_D \otimes \hat{A})^*$. It remains to mod out by the choice of the generator of $F\Lambda$, *i.e.* by D^*, to establish the desired one-to-one correspondence. $\square$

analysis of the proposition

4.2. Chaining the last proposition with the equivalences 3.6 and 2.13 furnishes a bijection between the set of isomorphism classes of $\mathcal{D}$-elliptic sheaves on $X_{\mathbf{C}}$ and the above double co-sets. This double co-set space inherits the structure of a rigid analytic space from the rigid analytic space Ω^d (see [3]). So, at least point-wise, $\mathcal{D}$-elliptic sheaves are classified by a rigid analytic space, but this set-theoretic result is not enough to give an analytic structure to the algebraic moduli spaces of Laumon, Rapoport and Stuhler.

analytic moduli

4.3. A close examination of the proof of proposition 4.1 shows that it establishes a slightly stronger result. In fact, it demonstrates that every analytic family of pairs (V, Λ) over a rigid analytic $\mathbf{C}$-space Y - the definition of such a family should be clear - results in an analytic map from Y to the double co-set space. In fact, also this stronger statement can be easily pulled through the chain of equivalences 3.6 and 2.13, using the results of [2]. The outcome is a procedure that assigns to every $\mathcal{D}$-elliptic sheaf on X_B for some $\mathbf{C}$-algebra B, a morphism of rigid analytic $\mathbf{C}$-spaces from $\mathrm{Spec}(B)^{\mathrm{an}}$ to the double co-set space of the proposition.

algebraic moduli

4.4. In [7] it is shown that when $I \neq A$, the functor that "maps" a $\mathbf{C}$-algebra B to the set of isomorphism classes of $\mathcal{D}$-elliptic sheaves on X_B is representable by a quasi-projective $\mathbf{C}$-scheme of dimension $d-1$, denoted by $\mathcal{E}\ell\ell_{X,\mathcal{D},I}$. Taking the quotient by a finite group yields a coarse moduli scheme $\mathcal{E}\ell\ell_{X,\mathcal{D},A}$ classifying $\mathcal{D}$-elliptic sheaves without level structure. All these varieties are shown to be smooth and under the condition that D be a division algebra they are shown to be complete in *loc.cit.*

relating algebraic and analytic moduli

4.5. Now the above considerations define a natural map of rigid analytic spaces

$$\mathcal{E}\ell\ell_{X,\mathcal{D},A}(\mathbf{C})^{\mathrm{an}} \to D^*\backslash \left[\Omega^d(\mathbf{C}) \times D^*(\mathbb{A}_A)/U\right],$$

which is a bijection on the sets of $\mathbf{C}$-valued points. Since both spaces are reduced, this has to be an isomorphism. This proves the fundamental case of the main theorem.

principal case of the theorem

4.6. **Theorem.** *There is a natural isomorphism of rigid analytic spaces*

$$\mathcal{E}\ell\ell_{X,\mathcal{D},A}(\mathbf{C})^{\mathrm{an}} \cong D^*\backslash \left[\Omega^d(\mathbf{C}) \times D^*(\mathbb{A}_A)/U\right].$$

free vs. projective

4.7. A triple $(\mathcal{E}_i, j, t)$ is said to be *free* if and only if the restrictions of the $\mathcal{E}_i$ to $X \times s$ are free for all geometric points s of the base. For example, when X is the projective line and ∞ is rational then all elliptic $\mathcal{D}$-sheaves are free. This notion readily translates to Anderson A-motives.

M being free then signifies M being free as $A \otimes \mathbf{C}$-module. Free $\mathcal{D}$-elliptic sheaves are classified by closed and open subspaces $\mathcal{E}\ell\ell^0_{X,\mathcal{D},I} \subset \mathcal{E}\ell\ell_{X,\mathcal{D},I}$. The above theorem quite easily generalises to the main result of this exposition.

main theorem

4.8. **Theorem.** *The analytification of the moduli spaces $\mathcal{E}\ell\ell^0$ and $\mathcal{E}\ell\ell$ is*

$$\mathcal{E}\ell\ell^0_{X,\mathcal{D},I}(\mathbf{C})^{\mathrm{an}} \cong G(I)\backslash\Omega^d(\mathbf{C})$$

and

$$\mathcal{E}\ell\ell_{X,\mathcal{D},I}(\mathbf{C})^{\mathrm{an}} \cong D^*\backslash\left[\Omega^d(\mathbf{C}) \times D^*(\mathbb{A}_A)/U(I)\right]$$

where $G(I) \subset O_D^*$ and $U(I) \subset U$ are the subgroups of elements that reduce to 1 modulo I and where D^* acts on $\Omega^d(\mathbf{C})$ by the choice of an isomorphism $D \otimes_F F_\infty \cong \mathrm{M}(d, F_\infty)$.

If $D = \mathrm{M}(d, F)$, then the above result boils down to the uniformisation of the Drinfel'd modular varieties M^d_I as established in [3].

quaternionic Shimura curves

4.9. **Example.** When O_D is a maximal order in a quaternion algebra over F that is unramified at ∞, then for every level I, the moduli space described above is a complete smooth curve over $\mathbf{C}$. By the above theorem, these spaces are quotients of $\Omega^2 \subset \mathbb{P}^1(\mathbf{C})$ by some subgroup G of $O_D^* \subset D \otimes_F \mathbf{C} \cong \mathrm{M}(2, \mathbf{C})$. It follows from the theory of Mumford curves (see *e.g.* [8], [4]) that the genus of this curve is the rank of G, *i.e.* the rank of the Abelianisation of G. It is in general however not straightforward to calculate the rank of a given G. Usually it is easier to find a larger group $H \supset G$ such that $H\backslash\Omega^2$ has genus zero, and to calculate the genus of $G\backslash\Omega^2$ from the ramification of $G\backslash\Omega^2 \to H\backslash\Omega^2 \cong \mathbb{P}^1$.

References

[1] Greg W. Anderson. t-motives. *Duke Math. J.*, 53(2):457–502, 1986.

[2] Gebhard Böckle and Urs Hartl. Uniformizable families of t-motives. *Trans. Amer. Math. Soc.*, 2005.

[3] В. Г. Дринфельд. Эллиптические модули. *Матем. Сб.*, 94(4):594–627, 1974.

[4] Lothar Gerritzen and Marius van der Put. *Schottky groups and Mumford curves*, volume 817 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[5] Thomas Hausberger. Uniformisation des variétés de Laumon-Rapoport-Stuhler et conjecture de Drinfeld-Carayol. *Ann. Inst. Fourier (Grenoble)*, 55(4):1285–1371, 2005.

[6] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[7] G. Laumon, M. Rapoport, and U. Stuhler. $\mathcal{D}$-elliptic sheaves and the Langlands correspondence. *Invent. Math.*, 113(2):217–338, 1993.

[8] David Mumford. An analytic construction of degenerating curves over complete local rings. *Compositio Math.*, 24:129–174, 1972.

E-mail address: `lenny@math.rug.nl`